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differs from the true root by a small quantity of the second order only. Since we can always take for x the nearest whole number to the root considered, h will be a proper fraction less than $\frac{1}{2}$, and the error committed will be less than

$$\frac{n-1}{8x};$$

that is, less than $\frac{5}{4}(n-1)$ units in the m^{th} place of decimals, m being the number of integral places in x.

The principal value of the method will be found in extending the limits of application of ordinary tables of roots. For example, to find

$$\sqrt[3]{3.141593}$$

we take from a table of cubes 1.46 as the first approximation. We find

$$3.141593 \div 1.46^2 = 1.47382;$$
$$\therefore \xi = 1.46461$$

within $2\frac{1}{2}$ units of the last place. The true root is 1.46459. The next approximation would give nine places, and so on.

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MR. JAMES GLAISHER'S FACTOR TABLES AND THE DISTRIBUTION OF PRIMES.

By Prof. W. W. Johnson, Annapolis, Md.

The publication of Mr. James Glaisher's *Factor Table for the Sixth Million** marks the completion of a work of extraordinary extent, and of the highest value in the theory of numbers, to which France, Germany, and England have contributed equal portions of the tireless zeal of the skillful calculator; namely, the tables which enable us to resolve into its prime factors every number less than 9,000,000. The present volume is noteworthy also as containing, in the form of an introduction of more than 100 quarto pages, a very able memoir upon the distribution of primes, embodying in tabular form the results of profound and laborious research.

The history of the factor tables is briefly as follows: The subject of the construction of extensive factor tables was agitated to a considerable extent during the latter part of the last century. Euler published a memoir in 1774 on the mode of construction of a factor table to extend to a million, and four years earlier, Lambert, in a work containing a description of a method of forming such a table, gave a folding sheet on which were given all the simple factors of numbers not divisible by 2, 3 or 5 from 1 to 10,200, and he urgently appeals to calculators in the hope that some one will extend this table to 102,000, or "if he will make himself an immortal name," to 1,020,000. But it was not until 1810 that a complete factor table of the first million appeared. This was the *Cribrum Arithmeticum* of Chernac, which gives all the prime factors of every number not divisible by 2, 3 or 5 up to 1,020,000. Chernac speaks of this as the labor, not of months, but of years. Three years later, Burckhardt published his table for the second million, or more strictly from the limit of Chernac to 2,028,000. This gives the least factor only, which obviously effects an enormous saving in bulk, while impairing the usefulness of the table only to a slight extent, the process being of course to divide by the tabular factor and enter the table for the first million with the quotient. To facilitate the divisions, a table of the first nine multiples of all primes up to 1423 was given on the first page. In 1816 Burckhardt published the continuation of this table to 3,036,000, and in 1817 he published the first million in a table uniform with his other volumes. He states that this table is based upon a comparison of a manuscript by Schenmark, in the possession of the library of the Institute, with Chernac, the latter being found correct in nearly every case of discrepancy. Burckhardt expresses his willingness to prepare the manuscript of the fourth, fifth, and sixth millions, if the sale of the three millions, which were also published in a single volume, shall appear to the publisher sufficiently favorable.

**Factor Table for the Sixth Million, containing the least factor of every number not divisible by* 2, 3 *or* 5 *between* 5,000,000 *and* 6,000,000. By James Glaisher, F. R. S. London: *Taylor and Francis*, 1883

In 1850 Gauss wrote to Zacharias Dase, who was possessed of a great natural talent for computation, to repeat, as he says, in writing what he had before told him by word of mouth. He refers in this letter to a manuscript of the fourth, fifth and sixth millions which had been presented by Crelle to the Berlin Academy, and adds that he doubts not that this will sooner or later be published; he therefore urges Dase to undertake the calculation of the next four millions. In 1860, through the support of the patrons of science in his native town, Hamburg, Dase was enabled to devote himself entirely to carrying out Gauss's project, but in 1861 he died suddenly, leaving the seventh million complete and the eighth nearly complete, as well as a great part of the work for the ninth and tenth millions. The seventh million volume, in the preface of which Gauss's letter is printed, was published in 1862 by the Committee of the Dase-Stiftung. Dr. Rosenberg, of Hamburg, undertook the continuation of the work, and the eighth million was published in 1863. The ninth million, described on the title page as by Zacharias Dase and completed by Dr. H. Rosenberg, followed in 1865; a short preface signed "Das Cómite der Dase-Stiftung" states that the tenth million is near completion. This million has, however, never been published.

In the report, for 1878, of the Committee on Mathematical Tables to the

British Association for the Advancement of Science, it is stated that, the committee deeming it very desirable that the gap of three millions between Burckhardt's and Dase's tables should be filled up, Prof. Cayley, the chairman of the committee, communicated early in 1877 with the Berlin Academy, with regard to the probability of the publication of the manuscript of Crelle, and learned that it had been examined and found to be so inaccurate that "the Academy was convinced that its publication would never be advisable." The calculation was then at once commenced by Mr. James Glaisher, a member of the committee, with the assistance of two computers. The table for the fourth million was published in 1879, with an extensive introduction largely historical, from which the above summary is chiefly taken. The fifth million was published in 1880, and the printing of the sixth million was commenced in that year, but the elaborate character of the introduction has delayed the appearance of this volume until 1883.

It has been learned that the manuscript of the tenth million, commenced by Dase and completed by Rosenberg, was presented in 1878 to the Berlin Academy by the widow of Dr. Rosenberg.

The arrangement of the tables is uniform for the nine millions. Owing to the rejection of the multiples of 2, 3 and 5, the last two figures of the numbers retained recur in a cycle of 300, there being 80 numbers in 300 retained. The quarto page admitting of 80 lines, the terminal figures of the argument are printed at the side of the page in three groups, each corresponding to 100 numbers. The columns are thus divided each into three sections, each section being headed by the two figures of the argument preceding those found at the side of the page. The entries in the columns are the least factors in the case of composite numbers and bars in the case of prime numbers. The mode of construction is due to Burckhardt, but being very briefly described by him is explained in some detail by Mr. Glaisher in the introduction to the fourth million. It is evident that, were all the numbers retained in the table, the multiples of any prime p would occur in certain diagonal straight lines, and therefore could be determined mechanically. The rejection of the multiples of 2, 3 and 5 dislocates this arrangement, but Burckhardt perceived that the arrangement of the multiples in the first p columns would still be repeated in the next p columns, and so on in each successive block of p columns. He therefore prepared his manuscript on sheets having the terminal figures printed at the side, and containing 77 columns, a multiple of 7 and 11 being chosen, so that the entries of the factors 7 and 11 would recur in the same position on each sheet. Thus the factors 7 and 11 could be printed upon the sheets, there being 880 entries of 7, and 480 entries of 11 on each sheet. This not only saves a great amount of labor, but ensures the accuracy of these 1360 entries on each sheet.

In like manner the multiples of 13 recur in the same positions in successive blocks of 13 columns. If a piece of paper of the size of 13 columns of the sheet, having the spaces which correspond to the multiples of 13 in the first 13 columns cut out, be laid upon these columns, the entry 13 may be made in all of the spaces of the sheet which appear through the holes unless they already contain a 7 or an 11. The screen thus constructed, which is called a "sieve," is then placed upon the next block of 13 columns and the entry 13 made through the holes, and so on. A sieve for 17 is next used in a similar manner, and the entry 17 made in those spaces which appear through the holes and are not already occupied by 7, 11 or 13. Sieves were thus constructed for all prime numbers up to 307. A little consideration will show that each sieve will have 80 holes, one in each of the 80 lines, and this was a useful check in their construction. In practice Mr. Glaisher found it more convenient to employ for the small numbers sheets consisting of several sieves in juxtaposition (thus the 13,s were entered by a sieve of 78 columns, containing 6 copies of the sieve), and for the large primes the sieves were constructed from several sheets which were not fastened together, but numbered and used successively, each sheet of the sieve, of course, in general overlapping two of the sheets of the table.

For primes greater than 307 another method was employed. It is obvious that every number for which a prime greater than 307 is the *least* factor, and within the limits of the table, is the product of two primes. To determine these multiples, in the case for example of 311, we have therefore to multiply 311 by successive prime numbers. The first multiple occurring in the fourth million was 9649×311 = 3,000,839; the series of primes commencing at 9649, is 9649, 9661, 9677, . . . , the differences being 12, 16, 2, . . . ; adding to 3,000,839 successively 12×311, 16×311, 2×311, . . . we have the multiples required. The multiples of the larger primes were thus prepared in books, from which the entries upon the sheets were made in the spaces indicated, after the sieves had been used. These spaces should, of course, all be found vacant, and this afforded a verification of the correct placing of the sieves.

The introduction to the fourth million contains a list of primes up to 30,341, with their differences.

Considered solely with reference to the frequency with which the results may be required, it might be thought that a factor table should not rank high among mathematical tables for utility, but this measure of comparative value is enormously increased, when we remember the extreme laboriousness with which the individual results would be obtained, when required, in the absence of a table. It would require, for example, a long day's work to prove that 5,999,993 is a prime, or even to resolve into factors 5,999,999, which has no factor less than 1000; in the latter case the verification would be simple enough, but in the former the whole calculation would have to be duplicated before we could put any confidence in the result. In fact the contrast between the difficulty of obtaining the individual results without a table and the comparative ease with which the table, as a whole, is constructed is most striking.

A portion of Burckhardt's introduction relates to the calculation of the logarithms of large numbers by means of a factor table in connection with such tables as Wolfram's table of the hyperbolic logarithms of primes up to 10,009 to 48 decimal places; and the introduction to the fourth million contains an example, the principle being to multiply by a factor whose logarithm is known, such that the product shall be within the range of the factor table, and shall on trial be found to differ by only one or two units from a composite number, x, all of whose prime factors are within the range of the logarithmic table. The logarithm is then obtained by the use of the formula

$$\log (x + a) = \log x + \frac{a}{x} - \tfrac{1}{2}\frac{a^2}{x^2} + \tfrac{1}{3}\frac{a^3}{x^3} - \text{etc.},$$

in which, of course, the object is to render $\frac{a}{x}$ as small as possible.

"But," to quote Mr. J. W. L. Glaisher (*Camb. Phil. Proc., Feb.* 11, 1878), "the real importance of an extended factor table lies in the fact that such a table is a fundamental one in the Theory of Numbers. The number of factors of a number, their sum, etc., are elements which enter into many results in the Theory of Numbers, and it is clear that, even on this account alone, it is desirable to have a table by means of which the resolution of high numbers into their factors is rendered practicable, so that suitable examples, verifications, etc., of such theorems may be obtainable. Also, conjectural theorems of this nature could not readily be tested without such tables. Again, the law of frequency of prime numbers has been the subject of analytical investigation, and theoretic formulæ giving the average frequency of primes, and the approximate number of primes between any given limits, have been obtained by Legendre, Tchebycheff, Hargreave, Riemann, and others, and it is essential to have the means of comparing the numbers given by these formulæ with those found by actually counting the primes."

Legendre published in 1808 his formula

$$\frac{x}{\log x - 1.08366}$$

for the number of primes inferior to x. The reasoning by which he arrives at the form

$$\frac{x}{A \log x - B}$$

is vague and unsatisfactory, and no account is given of the method by which the constants were determined. Legendre gives comparisons of the value of this formula with the number of primes counted, up to 400,000, from Vega's *Tabulæ.* Legendre's confidence in this formula was perhaps based upon the fact that, although the constant may have been determined from the value $x = 10{,}000$, the results were satisfactory up to 400,000. He says, "Il est impossible qu' une formule représente plus fidèlement une série de nombres d' une aussi grande étendue et sujette necessairement à de fréquentes anomalies." Subsequently Legendre extended his comparisons to 1,000,000 by means of Chernac's *Cribrum* and the results were still satisfactory.

In 1849 Gauss refers, in a letter to Encke, to the value of the constant in Legendre's formula. He had, with the assistance of Goldschmidt, made an enumeration of primes from Burckhardt's three millions. He gives comparisons with Legendre's formula, and calculates the values of the constant which must be used in order to bring the formula into agreement with the numbers counted for $x = 500{,}000$, 1,000,000, etc., to 3,000,000. It appears that the constant must be decreased as x increases. It was subsequently proved by Tchebycheff that the constant must be unity when x is infinite. Gauss also gave comparisons with the logarithmic integral

$$\mathrm{li}\, x = \int_0 \frac{dx}{\log x}$$

Tchebycheff in 1848 arrived independently at the formula

$$\int_2 \frac{dx}{\log x}$$

and Hargreave in the following year represented the number of primes between x and x' by

$$\int_x^{x'} \frac{dx}{\log x}$$

In 1872, Mr. J. W. L. Glaisher, in a paper read before the British Association, gave the result of enumerations made from Burckhardt's and Dase's Tables, and comparisons, throughout the second and ninth millions, with the differences of the values of the integral li x. The enumerations of Gauss were found to be very inaccurate. In 1876 and 1877, in communications to the Cambridge Philosophical Society, Mr. Glaisher, having carefully revised the enumerations, gives the results for the six millions which had then been published, and states his purpose of comparing the results with Legendre's formula and with the logarithmic integral throughout the six millions. These enumerations were extended to the fourth and fifth millions, and the results appeared in the introductions to those volumes; and the introduction to the sixth million contains tables giving the number of primes in each successive group of 1,000 numbers from 0 to 9,000,000. Also in each group of 10,000 numbers, and in groups of 100,-

000, of 250,000, of 500,000, and in each million. The irregularity of the distribution of primes is strikingly shown by the differences which are added in the tables where the groups contain 100,000 numbers or more; "not even in the table where the groups contain 500,000 numbers do the numbers of primes in the successive groups always decrease, there being an increase in one instance."

The results for the complete millions are as follows:

n	Number of primes in the nth million.	Difference.	Number of primes inferior to n millions.
1	78,499	—8066	78,499
2	70,433	—2548	148,932
3	67,885	—1556	216,817
4	66,329	— 960	283,146
5	65,369	—1033	348,515
6	64,336	— 537	412,851
7	63,799	— 641	476,650
8	63,158	— 398	539,808
9	62,760		602,568

In 1859 Riemann communicated to the Berlin Academy a paper (which appears at page 136 of his collected works) in which he investigates the law of frequency of primes, and finds for the non-periodic part of the expression for the number of primes inferior to x the formula

$$\mathrm{li}\, x - \tfrac{1}{2}\,\mathrm{li}\, x^{\frac{1}{2}} - \tfrac{1}{3}\,\mathrm{li}\, x^{\frac{1}{3}} + \tfrac{1}{6}\,\mathrm{li}\, x^{\frac{1}{6}} - \tfrac{1}{7}\,\mathrm{li}\, x^{\frac{1}{7}} + \text{etc.},$$

in which the terms are of the form $\pm \frac{1}{n}\,\mathrm{li}\, x^{\frac{1}{n}}$, where n is the product of primes all different, and the sign is + or —, according as the number of primes is even or odd. To this memoir the late Professor H. J. S. Smith referred in 1876 as containing "the only investigation of the asymptotic frequency of the primes which can be regarded as rigorous."

The introduction to the sixth million contains the complete comparisons, not only with Legendre's formula, and with li x, but also with Riemann's formula at intervals of 50,000 for the whole nine millions. The tables also give the deviations of the formulas from the actual numbers of primes. These deviations show the great superiority of Riemann's formula, which is found to be more accurate than Legendre's even for the smaller values of x, and to give satisfactory results for the whole nine millions. Legendre's formula, on the other hand, commences at 650,000 to give results which are, with a single exception (at 1,200,000) always in excess, the excess amounting to 391 at 8,450,000 and to 282 at 9,000,000. The formula li x gives results largely in excess, even for the smaller values of x, and steadily diverges, but not so rapidly as Legendre's formula; so that it is more accurate than the latter when x exceeds five millions. The mean deviations derived from the table are, for Riemann's formula, — 9, for li x, + 163, and for Legendre's formula, + 171.

To exhibit the deviations more clearly to the eye, Mr. Glaisher has given a diagram in which successive values of the mean of three deviations in the table are taken as ordinates corresponding to the middle values of x as abscissas. These "deviation curves" are, of course, extremely irregular, owing to the anomalous distribution of the primes; but, while the other curves tend steadily upward, Riemann's curve shows no tendency to cumulative error; in fact it cuts the axis 18 times. Had the actual values of the deviations at intervals of 50,000 been constructed as ordinates, the curve would have been still more irregular, but it would have cut or touched the axis 51 times within the nine millions, the last time at 8,500,000; the deviation at 9,000,000 is — 132.

It is possible to determine the number of primes, for particular values of x, without actually constructing the factor table. The number of multiples of p, including p itself, inferior to x is $E\left(\frac{x}{p}\right)$, the greatest integer which does not exceed $\frac{x}{p}$. Taking account of the numbers divisible by both p_1 and p_2, the number divisible by either p_1 or p_2 is $E\left(\frac{x}{p_1}\right) + E\left(\frac{x}{p_2}\right) - E\left(\frac{x}{p_1 p_2}\right)$; and in general the number divisible by one of the primes $p_1, p_2, \ldots p_n$ is

$$\Sigma E\left(\frac{x}{p_1}\right) - \Sigma E\left(\frac{x}{p_1 p_2}\right) + \Sigma E\left(\frac{x}{p_1 p_2 p_3}\right) - \text{etc.}$$

The employment of this formula it will be noticed is equivalent to calculating the number of entries of one of the primes $p_1, \ldots p_n$ that would be made by the "sieve" method, if a factor table were to be constructed to extend to x, but in the case of primes greater than $\sqrt[3]{x}$ which occur as least factors of two-factor numbers only, within the limit of the table, the number of times p would be entered may be determined by a method analogous to the "multiple" method described above, *i. e.*, it is only necessary to count the primes between p and $\frac{x}{p}$. Thus the whole number of entries may be determined, and by subtraction the number of primes. The process is, of course, very laborious, but Meissel in 1870 and 1871 employed it to calculate the number of primes inferior to one million, to ten millions, and to one hundred millions. Meissel's number for one million was exactly confirmed by the actual count from Burckhardt's table, and Mr. Glaisher regards his results for ten millions and one hundred millions, namely, 664,580 and 5,761,461, as entitled to be accepted with confidence. Accordingly the values of the several formulæ were calculated for $x = 10{,}000{,}000$ and for $x = 100{,}000{,}000$. At ten millions, Riemann's formula gives an excess of 87, so that the deviation curve crosses the axis at least once between nine and ten millions; the deviation of li x is 338, and that of Legendre's formula is 560. At one hundred millions Riemann's deviation is only 90, that of li x is

748, and that of Legendre's formula is 6543. The accuracy of Riemann's formula at this point is extraordinary, the error being less than $\frac{1}{60,000}$ of the actual number of primes.

The introduction to the sixth million, which contains many important and interesting results besides those briefly summarized in this review, concludes with a discussion of the mode of calculating the values of li x and of Riemann's formula.

A mathematical table of this magnitude and importance may well be reckoned one of the "long results of time;" and England may be justly proud that such a work has not only been completed by one of her own mathematicians, but furnished with so worthy a crown.

EXERCISES.

1

A HORIZONTAL WIND blows against a hemispherical dome of radius R'. The pressure exerted by the wind on a plane surface normal to its direction is P pounds to the square foot; on a surface oblique to its direction the pressure exerted is normal, but is reduced in the ratio (Poncelet, *Mécanique Industrielle*, 403)

$$1 : 1 + \tfrac{1}{2} \tan^2 i,$$

where i is the angle of incidence. It is required to find the magnitudes and points of application of the horizontal and vertical components of the resultant wind-pressure. [*W. M. Thornton.*]

2

THREE closely connected tanks, T_1, T_2, T_3, contain Q_1 gallons of water, Q_2 gallons of vinegar, Q_3 gallons of brandy, respectively. A flow is set up from T_1 through T_2 to T_3 and back to T_1 at the rate of 1 gallon per second. The liquids are assumed to mix instantaneously, and the lengths of the connecting pipes are neglected. Show how to calculate the amount of water in each tank at the end of T seconds. [*De Volson Wood.*]

3

To find the radius of a sphere, which being let fall into a given conical wine-glass, full of water, will displace the maximum or minimum quantity of water. [*Benjamin Alvord.*]

4

ON one of the bounding radii of a given quadrant, radius r, a semicircle is drawn, radius $\frac{1}{2}r$, the semicircle being within the quadrant. Find the average area of the circle touching the other bounding radius of the quadrant and the arc of the semicircle. [*Artemas Martin.*]